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3 Persons, 2 Cuts: A Maximin Envy-Free and a Maximally Equitable Cake-Cutting Algorithm

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Abstract

We describe a 3-person, 2-cut envy-free cake-cutting algorithm, inspired by a continuous moving-knife procedure, that does not require that the players continuously move knifes across the cake. By having the players submit their value functions over the cake to a referee—rather than move knives according to these functions—the referee can ensure that the division is not only envy-free but also maximin. In addition, the referee can use the value functions to find a maximally equitable division, whereby the players receive equally valued shares that are maximal, but this allocation may not be envy-free.

1. Introduction

Everyone knows the cake-cutting procedure, "I cut, you choose:" Person *A* (assumed male) divides the cake, which may be any heterogeneous divisible good, into two equal parts—in terms of his preferences—and person *B* (assumed female) chooses the piece that she thinks is at least as valuable as the other piece.[1] Thereby each person believes he or she received a piece at least as valuable as the piece obtained by the other person. This division is *envy-free*, because *A* and *B* do not envy the other person for having obtained a more valuable piece.

But can this procedure be extended to more than two persons? John Selfridge and John Conway, circa 1960, independently proposed an extension for dividing a cake into envy-free portions for three persons, whom we will refer to as players. It was never published but is described in Brams and Taylor (1996, pp. 116-120). Instead of using the minimal two cuts to divide a cake into three pieces, however, it requires up to five cuts. Moreover, the portions that players *A*, *B*, and *C* receive may not be connected (or contiguous)—one or more of them may get pieces from different parts of the cake (e.g., both ends) rather than a single connected piece.

Twenty years later Stromquist (1980) proposed an algorithm that gives players connected pieces, using two cuts for three persons, but it was not discrete: It used four simultaneously moving knives, requiring *A*, *B*, or *C* to call "stop" when their moving knives and the moving knife of a referee created a tie between different pieces of the cake.

Some 25 years after this discovery, Barbanel and Brams (2004) found a much simpler 3-person moving-knife algorithm that required a player, or a player and a referee, to move two parallel knives simultaneously. By "squeezing" a piece desired by two players until it creates a tie with another piece for one of the players, it enables all three players to obtain at least a tied-for-most-valuable piece.

They also showed how this algorithm could be extended to four players, but instead of using only the minimal three cuts, it required up to five. This is fewer than the

[1] We thank Daniel Brik Laufer for valuable research assistance.

11 cuts that an earlier 4-person moving-knife algorithm of Brams, Taylor, and Zwicker (1997) required.

Besides 3-person and 4-person algorithms that use relatively few cuts, envy-free algorithms that work for any number *n* of players have been developed. These include an *n*-person moving-knife algorithm (Brams and Taylor, 1995), which uses a finite number of cuts, but this number is *unbounded*—an upper bound cannot be specified independently of the preferences of the players—and an algorithm based on Sperner's Lemma (Su, 1999), which requires convergence to an exact division so also is unbounded.[2]

Aziz and Mackenzie (2017) provided an algorithm that has two advantages: It does not depend on continuously moving knives or convergence, and it is finite and bounded. Its disadvantage is that its bounds are presently extremely large, making it, like Brams and Taylor's algorithm, computationally complex; in particular, it is not solvable in polynomial time. Recently, it was modified and extended to chore division (Dehghani et al., 2018), and a simplified 4-person discrete cake-cutting algorithm was found for four players (Amanatidis et al., 2018).

In this paper, we return to the 3-person cake-cutting problem but provide an envy-free algorithm that

- does not require moving knives (unlike Stromquist's, Brams, Taylor, and Zwicker's, and Brams and Barbanel's algorithms);
- requires only the minimal two cuts and so gives connected pieces (unlike Selfridge and Conway's algorithm);

The algorithm is not computationally complex for three players, but it does require a number of steps that we specify later.

After describing this envy-free algorithm, we show how it can be extended to give an envy-free allocation that is

- *maximin envy-free*—it maximizes the minimum value that any player receives from two cuts.

We also show how to find an allocation that is

- *maximally equitable*—all players receive equally valued shares that are

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[2] The numerical solutions we give for the two examples in section 4, in which the players' value functions are piecewise linear, also are not exact. But these examples have exact, or closed-form, solutions that can be expressed in terms of radicals—as do the solutions of any polynomial equations of 4th degree or less (those for the examples are quadratic)—so they do not require convergence to an exact solution. Exact solutions for the examples in section 4 are given by the Wolfram Equation Solver (2018).

maximal.

Both kinds of allocations are *efficient (Pareto-optimal):* There is no other allocation in which all the players do at least as well and one or more does better, at least for 2-cut allocations.

In the case of envy-freeness, efficiency follows from the fact that any envy-free allocation that uses the minimal n–1 cuts is efficient (Gale, 1993; Brams and Taylor, 1996, pp. 149-151); maximin envy-freeness additionally ensures that the worst-off player does as well as possible. In the case of maximal equitability, an allocation that gives all players equally valued shares that are as great as possible cannot give any player more without giving another player less, so it is also efficient. Although it is well known that there exists an envy-free allocation that is also equitable (Neyman, 1946), it need not be efficient; for more on the efficiency of envy-free allocations—but not algorithms to find them—see Barbanel (2005).

2. Submit Marks and, if Necessary, Value Functions

We make the following assumptions:

1. The cake is defined by the interval [0,1], and a division of the cake is a partition, indicated by points on the interval, into three connected subintervals that together comprise the entire cake.

2. Each player has a probability density function (pdf) over the cake, called a *value function*, whose measure is assumed to be nonatomic and absolutely continuous—that is, the pdf is continuous on [0,1], takes only non-negative values, is positive somewhere on each subinterval, and has (Riemann) integral equal to 1. (The later examples in this paper of pdfs that are piecewise linear makes the integrals simple to evaluate, but for functions for which one cannot easily determine an indefinite integral, this may not be the case.)

How our algorithm differs from the aforementioned algorithms is that each player—independently of the other players—begins by placing marks on the interval that defines the cake. These marks indicate the two points that trisect the cake into three equally valuable pieces for each player. Later the players provide, if necessary, information on their value functions, which we discuss in section 3.

Let A's 1/3 and 2/3 marks be a_1 and a_2, B's marks be b_1 and b_2, and C's marks be c_1 and c_2. Thus, A values the subintervals $[0, a_1]$, $[a_1, a_2]$, and $[a_2, 1]$ at 1/3 each, and analogous marks by B and C define 1/3 subintervals for each player. (Because the players' measures are nonatomic, all intervals can be assumed to be closed—the endpoints do not matter.) To avoid ties, we assume that each player's marks on [0, 1] are different from the other two players' marks.

Depending on where the players place their 1/3 and 2/3 marks, the algorithm then determines, starting from 0 (on the left side of the cake), which player's first mark (a_1, b_1, or c_1) is closest to 0 (for simplicity, assume there are no ties). Assume that player is A, so A is the only player to value $[0, a_1]$ at 1/3, whereas B and C value it at less than 1/3.

We begin by considering three cases, in each of which we assume that cuts are made at A's marks, a_1 and a_2 (underscored in the diagrams that follow). Assume that A receives the left piece $[0, a_1]$, which only A values at 1/3.

We can then determine those cases in which B and C most value different pieces, which may be either $[a_1, a_2]$ or $[a_2, 1]$, or the same piece. These cases are only illustrative, not exhaustive, of how the players' marks may (or may not) lead to an immediate envy-free division. We will later discuss cases in which this determination cannot be made from the 1/3 marks alone.

Case 1: B and C most value different pieces.

0----------$\underline{a_1}$-----b_1-----b_2-----$\underline{a_2}$-----c_1-----c_2----------1

B values the middle piece $[a_1, a_2]$, whose points are underscored, at more than 1/3 and the left and right pieces at less than 1/3, and C values the right piece $[a_2, 1]$ at more than 2/3. If B and C receive their most preferred pieces, and A receives the left piece, the allocation is envy-free.

Case 2: B and C most value the middle piece.

0----------$\underline{a_1}$-----b_1-----c_1-----b_2-----c_2-----$\underline{a_2}$----------1

Because B and C both value the middle piece $[a_1, a_2]$ at more than 1/3 and the left and right pieces at less than 1/3, whoever of B or C receives the middle piece will be envied by the other player.

Case 3: B and C most value the right piece.

0----------$\underline{a_1}$-----$\underline{a_2}$-----b_1-----c_1-----b_2-----c_2----------1

Because B and C both value the right piece $[a_2, 1]$ at more than 2/3 and the left and middle pieces at less than 1/3, whoever of B or C receives the right piece will be envied by the other player.

Clearly, making cuts at a_1 and a_2 in Cases 2 and 3 does not produce an envy-free allocation, whichever of B or C gets the middle piece in Case 2 or the right piece in Case 3. There are other cases in which making cuts at a_1 and a_2, and giving $[0, a_1]$ to A, may or may not allow for an envy-free allocation. Consider, for example, the following placement of marks by the players:

0---------$\underline{a_1}$-----b_1-----c_1-----b_2-----$\underline{a_2}$-----c_2---------1

If *A* receives $[0, a_1]$ and *B* receives $[a_1, a_2]$, they will not envy each other: A values $[0, a_1]$ at 1/3, the same as the other two pieces; *B* values $[a_1, a_2]$ at more than 1/3, and the left and right pieces at less than 1/3.

But it is unclear whether giving the right piece $[a_2, 1]$ to *C* will cause *C* to be envious. While this piece is worth more than 1/3 to *C*, it is possible that *C* values the middle piece $[a_1, a_2]$ more than this piece, based on *C*'s marks. In sum, we do not have sufficient information from the players' marks alone to say whether *C* values the right piece more than the middle piece.

The algorithm described in section 3 is applicable both to the unambiguous cases of envy (Cases 2 and 3) and ambiguous cases, like that just described, when it is uncertain whether making cuts at a_1 and a_2 produces an envy-free division. This algorithm is related to the "squeezing procedure" (Barbanel and Brams, 2004), in which *A* continuously moves two parallel knives (a referee may move one, and *A* the other in response), reducing a piece that both *B* and *C* prefer until one of these players indicates that this piece ties with one of *A*'s continuously increasing pieces. In Case 2, the squeezing by *A* is inward from both sides of the middle piece $[a_1, a_2]$; in Case 3, the squeezing is from the left side of $[a_2, 1]$. In either case, *A* squeezes one piece with the two parallel moving knives so that the two other pieces are equally valued by *A*.

But unlike the squeezing procedure, our algorithm determines directly—without using moving knives—the two points on [0, 1] at which to cut the cake so that a tie occurs with one of the two enlarged pieces that *A* values equally. (The Intermediate-Value Theorem and continuity ensure that a tie must occur.) It then assigns the tied enlarged piece to the player (say, *B*) for whom the squeezed piece ties; *C* gets the squeezed piece, which it still thinks is the most valuable; and *A* gets the remaining enlarged piece, which it thinks ties with the other enlarged piece and is preferable to the squeezed piece. Thereby, all the players receive at least tied-for-most valuable pieces, rendering the resulting allocation envy-free.

Unless the marks are as illustrated in Case 1, however, wherein each player most values a different piece, there is insufficient information to determine an envy-free allocation. When this is the case, each player independently submits to a referee its value function.

From this information, the referee determines, in a manner to be specified, how much the middle piece (Case 2) or the right piece (Case 3) must be squeezed to create a tie with one of the other two pieces. If the situation is ambiguous, the pdfs allow one to determine whether cuts at a_1 and a_2 immediately yield an envy-free allocation, as in Case 1. If they do not, the reduction required of either the middle piece or the right piece is found by solving two integral equations simultaneously, obviating the need to use moving knives.

Stromquist (2008) defined a "finite protocol" to be one in which the players make finitely many marks in order to determine an envy-free allocation. He showed that no finite protocol yields an envy-free allocation for three or more players. Given his definition, our algorithm is not finite, because it may use information on the continuous pdfs, not just the players' 1/3 and 2/3 marks, to determine an envy-free allocation.

But our algorithm does *not* involve the continuous choices required of moving-knife procedures. In place of making finitely many marks, we require (except in Case 1) that for an allocation to be envy-free, the players submit their pdfs to a referee. Then the referee makes decisions—specifically, when there is a tie in the value of two pieces—but this is based on solving equations, not on the continuous movement of knives.

Ianovski (2012) argued that "a moving knife protocol is certainly less than an effective procedure in the algorithmic sense," because, as Dehghani et al. (2018) pointed out, "the continuous movement of a knife cannot be captured by any finite protocol." This difficulty disappears, however, by introducing a referee, who uses the value functions of the players to find cutpoints (in a manner to be described in section 4).

Thereby the players do not need to make decisions about the stoppage of continuously moving knives but only about their inputs (their 1/3 marks and, if necessary, their pdfs). The referee uses this information to do the rest, comparing the values that players attach to different pieces and indicating when they are equal. His or her decisions, using the algorithms described in section 4, are purely mechanical.

If the referee cannot determine an envy-free allocation from the players' marks alone, then the players must submit their pdfs. Although the pdfs provide richer information on player preferences than do the marks, the *process* of finding an envy-free allocation, via our algorithms, still terminates after a finite number of steps.

The envy-free algorithm specifies the steps—at most five—to process this information and determine an envy-free allocation; an additional step is needed to find a maximin envy-free allocation. A maximally equitable allocation may require that different sets of equations be solved simultaneously.

The algorithms that yield maximin envy-free and maximally equitable allocations provide exact solutions (if the players' value functions are sufficiently simple; see note 1) without having physically to move knives and say at what point two pieces have the same value (the referee does this by solving equations). Since the players do not need to make an infinite number of continuous choices over time, the algorithms offer an immediate calculable solution rather than an extended physical solution.

Because the players submit information about their marks and, if necessary, their value functions independently, they cannot use information about the other players' marks and value functions to manipulate the algorithm to their advantage. But to ensure that the proper calculations are made, it would be advisable to specify a checking process for verifying their correctness.

3. Envy-Free and Equitable Algorithms

Our envy-free algorithm is implemented in the five steps specified next, where the value functions (i.e., pdfs) of *A*, *B*, and *C* are, respectively, $v_A(x)$, $v_B(x)$, and $v_C(x)$. Later we will specify the additional steps needed to find a maximin envy-free and a maximally equitable allocation.

The latter allocations cannot be found by moving-knife procedures, which assume that players know only their own value functions, whereas the referee in our setup knows the value functions of all the players. Our algorithms, therefore, afford benefits that moving-knife procedures do not by finding envy-free and equitable allocations that are maximal.

Envy-Freeness

Step 1. Assume *A*'s first mark, a_1, is the closest to 0. A referee determines whether cutpoints at a_1 and a_2 yield an envy-free allocation, whereby *A* gets $[0, a_1]$ and *B* and *C* most value different pieces—either $[a_1, a_2]$ or $[a_2, 1]$—as illustrated by Case 1. If yes, then *B* and *C* get their more valued pieces, which concludes the algorithm.

Step 2. If it is indeterminate whether cuts at a_1 and a_2 yield an envy-free allocation, the players submit their pdfs over [0,1]. A referee calculates the value that each player obtains from the different pieces, defined by the marks at a_1 and a_2. If *B* and *C* most value different pieces—either $[a_1, a_2]$ or $[a_2, 1]$—each receives the piece that it values more, and *A* receives $[0, a_1]$. This concludes the algorithm.

Step 3. If both *B* and *C* most value the middle piece, go to Step 4; if both *B* and *C* most value the right piece, go to Step 5. In each case, calculate the cutpoints *a* and *b* that yield an envy-free allocation, as given in Step 4 or Step 5.

Step 4 (middle piece squeezed by *A*'s equally expanding pieces on the left and right). Determine the upper and lower limits of integration, *a* and *b*, by solving simultaneously two pairs of integral equations, the first pair given by (1a) and the second pair given by (1b):

$$\int_0^a v_A(x)dx = \int_b^1 v_A(x)dx \text{ and } \int_a^b v_B(x)dx = \int_0^a v_B(x)dx \tag{1a}$$

$$\int_0^a v_A(x)dx = \int_b^1 v_A(x)dx \text{ and } \int_a^b v_B(x)dx = \int_b^1 v_B(x)dx\,. \tag{1b}$$

The equations to the left of "and" of (1a) and of (1b), which are the same, ensure that the value of the left and right pieces for *A* (the squeezer) are equal. The equations on the right of (1a) and (1b) ensure that the value of the middle piece (between *a* and *b*) for *B* equals

- its value of the left piece (between 0 and a), given by (1a); or
- its value of the right piece (between b and 1), given by (1b).

The referee chooses the piece—either left or right—for which the solution to the equations of (1a) or (1b) gives the larger value of $\int_a^b v_B(x)dx$. This is the piece that ties first with the middle piece under the moving-knife procedure, because it is reduced less than would be the other piece when it ties with the middle piece.

Next, determine the upper and lower limits of integration, a and b, by solving simultaneously the two sets of integral equations, analogous to (1a) and (1b), but in which v_C is substituted for v_B:

$$\int_0^a v_A(x)dx = \int_b^1 v_A(x)dx \text{ and } \int_a^b v_C(x)dx = \int_0^a v_C(x)dx \qquad (2a)$$

$$\int_0^a v_A(x)dx = \int_b^1 v_A(x)dx \text{ and } \int_a^b v_C(x)dx = \int_b^1 v_C(x)dx\ . \qquad (2b)$$

The referee chooses the piece—either left or right—for which the solution to the equations of (2a) or (2b) for a and b gives the larger value of $\int_a^b v_C(x)dx$. This is the piece that ties first with the middle piece under the moving-knife procedure, because it is reduced less than would be the other piece when it ties with the middle piece.

Compare the larger values given by the solutions of (1a) and (1b), and by the solutions of (2a) and (2b), and choose that which is greater (in the case of a tie, either can be chosen). This determines whether B or C is the first to receive an enlarged piece—either on the left or the right—that ties with the diminished middle piece. Assume it is B (C) that receives the enlarged piece when there is a tie; then C (B) will receive the diminished middle piece, which it still thinks is the most valuable; and A will receive the remaining enlarged piece on the left or the right. This concludes the algorithm.

Step 5 (right piece squeezed by *A*'s equally expanding pieces on the left). Determine the upper and lower limits of integration, a and b, by solving simultaneously two pairs of integral equations, the first given by (3a) and the second given by (3b):

$$\int_0^a v_A(x)dx = \int_a^b v_A(x)dx \text{ and } \int_b^1 v_B(x)dx = \int_0^a v_B(x)dx \qquad (3a)$$

$$\int_0^a v_A(x)dx = \int_a^b v_A(x)dx \text{ and } \int_b^1 v_B(x)dx = \int_a^b v_B(x)dx\ . \qquad (3b)$$

The equations to the left of "and" of (3a) and of (3b), which are the same, ensure that the value of the left and middle pieces of *A* (the squeezer) are equal. The second equations on the right of (3a) and (3b) ensure that the value of the right piece (between *b* and 1) for *B* equals

- the value of the left piece (between 0 and *a*), given by (3a); or
- the value of the middle piece (between *a* and *b*), given by (3b).

The referee chooses the piece—either left or right—for which the solution to the equations of (3a) or (3b) for *a* and *b* gives the larger value of $\int_b^1 v_B(x)dx$. This is the piece that ties first with the right piece under the moving-knife procedure, because it is reduced less than would be the other piece when it ties with the middle piece.

Next, determine the upper and lower limits of integration, *a* and *b*, by solving simultaneously the two sets of integral equations, analogous to (3a) and (3b), but in which v_C is substituted for v_B:

$$\int_0^a v_A(x)dx = \int_a^b v_A(x)dx \text{ and } \int_b^1 v_C(x)dx = \int_0^a v_C(x)dx \tag{4a}$$

$$\int_0^a v_A(x)dx = \int_a^b v_A(x)dx \text{ and } \int_b^1 v_C(x)dx = \int_a^b v_C(x)dx . \tag{4b}$$

The referee chooses the piece—either left or right—for which the solution to the equations of (4a) or (4b) for *a* and *b* gives the larger value of $\int_b^1 v_C(x)dx$. This is the piece that ties first with the right piece under the moving-knife procedure, because it is reduced less than would be the other piece when it ties with the middle piece.

Compare the larger values given by the solutions of (3a) and (3b), and by the solutions of (4a) and (4b), and choose that which is greater (in the case of a tie, either may be chosen). This determines whether *B* or *C* is the first to receive an enlarged piece—on the left or in the middle—that ties with the diminished right piece. Assume it is *B* (*C*) that receives the enlarged piece; then *C* (*B*) will receive the diminished right piece, which it still thinks is the most valuable; and *A* will receive the remaining enlarged piece on the left or in the middle. This concludes the envy-free algorithm. ■

Maximin Envy-Freeness

While the envy-free algorithm, via either Step 4 or Step 5, always finds cutpoints that yield an envy-free allocation, this allocation may not be maximin. However, it will be if the squeezed piece from Step 4 (middle piece) or Step 5 (right piece) is the least-valued piece, because it cannot be made more valuable without creating envy.

But if this is not the case, the squeezed piece that created the tie and an envy-free allocation can be squeezed further to create a second tie for the player who was not squeezed out initially. The maximin allocation will be the allocation between the first tie and the second tie that gives each player a different piece and maximizes the worst-off player's value. We will illustrate how to find this allocation in Examples 1 and 2 in section 4.[3]

Maximal equitability

Solve simultaneously the equations that equalize the players' shares, as they value them, by setting them equal to each other. If there is more than one solution, the referee chooses the one that maximizes the equal value that all players receive, which is efficient (Pareto-optimal).

In section 4, we show in both Examples 1 and 2 that the maximal equitable allocation need not be envy-free, much less maximally envy-free. These examples illustrate the possible conflict between maximal equitability and maximal envy-freeness, whose allocations give shares to each player that differ by several percentage points in these examples.

4. Two Examples

Example 1

Assume that A and B have the following piecewise linear value functions that are symmetric and V-shaped:

$$v_A(x) = \begin{cases} -4x + 2 \text{ for } x \in [0,\ 1/2] \\ 4x - 2 \text{ for } x \in [1/2,\ 1] \end{cases}$$

$$v_B(x) = \begin{cases} -2x + 3/2 \text{ for } x \in [0,\ 1/2] \\ 2x - 1/2 \text{ for } x \in [1/2,\ 1]. \end{cases}$$

Whereas both functions have maxima at $x = 0$ and $x = 1$ and a minimum at $x = 1/2$, A's function is steeper (higher maximum, lower minimum) than B's, as illustrated in Figure 1. In addition, suppose that a third player, C, has a uniform value function, $v_C(x) = 1$, for $x \in [0, 1]$.

[3] If no squeezing of pieces is required, as in Case 1 and some ambiguous cases in section 3, then a maximin envy-free allocation can be obtained not by starting from the envy-free allocation given by the 1/3 marks but instead by applying the maximin envy-free algorithm from scratch.

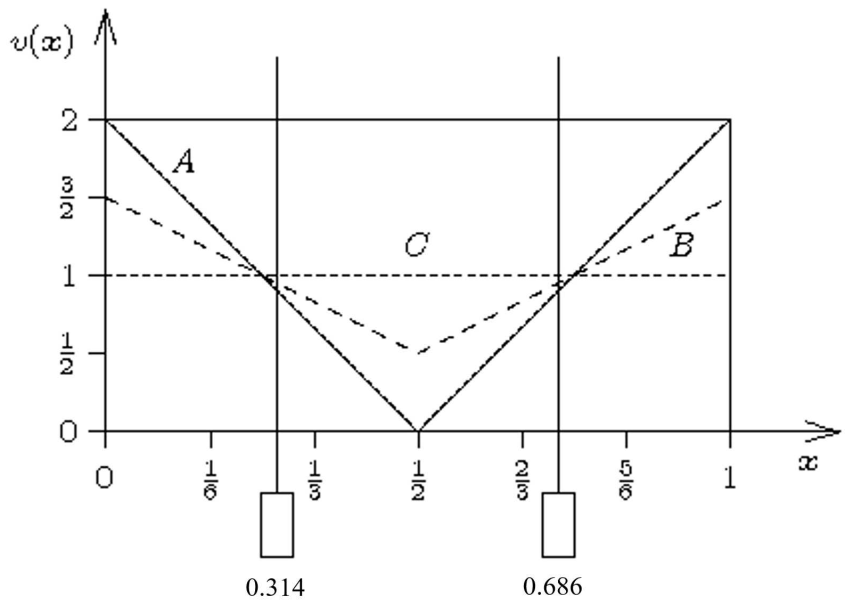

Figure 1. Value Functions *v*(*x*) of *A*, *B*, and *C* in Example 1 and Maximin Envy-Free Cutpoints

Before applying the algorithm, it is apparent that an envy-free allocation of the cake will be one in which *A* gets the portion to the left of *x*, *B* the portion to the right of 1–*x* (*A* and *B* could be interchanged), and *C* the portion in the middle. In Figure 1, we give the *maximin envy-free cutpoints*, represented by the "knives" below the graph, that show that *A* and *B* receive equal-length pieces between 0 and 0.314, and 0.686 and 1. If they were not equal, the player whose portion is shorter would envy the player whose portion is longer. Moreover, the distance separating the cutpoints must be at least 1/3 (it is 0.372); otherwise, *C* will envy *A* and *B* for getting pieces that it considers worth more than 1/3.

We will spell out shortly how we obtained the values for *a* and *b* that give the maximin envy-free cutpoints as well as cutpoints for both the envy-free and equitable allocations in Example 1. First, however, note that the 1/3 and 2/3 marks of the three players will appear as follows (though not to scale), where c_1 will be at $x = 1/3$ and c_2 at $x = 2/3$:

0-----$\underline{a_1}$-----b_1-----c_1------------------c_2-----b_2-----$\underline{a_2}$-----1.

Clearly, a_1 precedes b_1 and c_1 on the left, so we begin by asking whether *A*'s trisection yields an envy-free allocation. In Example 1, the marks show that both *B* and *C* prefer the middle piece $[a_1, a_2]$ to the right piece $[a_2, 1]$ and the left piece $[0, a_1]$.

If we were applying the squeezing procedure, the middle piece $[a_1, a_2]$ would be squeezed from the left and right by *A*, which corresponds to Step 4 of the algorithm. The first player whose valuation of the squeezed middle piece ties with the left and right pieces will be B, who places more value toward the endpoints of $x = 0$ and $x = 1$ than C does.

Both equations (1a) and (1b) are applicable to finding a and b when a tie occurs—for the first time with B—between the middle piece and the left and right pieces. Equation (5a) below ensures that the cutpoints are equidistant from 0 and 1 as A squeezes equally from the left and right, and equation (5b) ensures that B's middle piece, whose value is given by the left side of (5b), ties with its right piece, whose value is given by the right side of (5b). Thereby B can obtain this piece (it could also be the left piece because of the symmetry of the value functions about $x = 1/2$):

$$\int_0^a (-4x+2)\,dx = \int_b^1 (4x-2)\,dx \tag{5a}$$

$$\int_a^{1/2} (-2x+3/2)\,dx + \int_{1/2}^b (2x-1/2)\,dx = \int_0^a (-2x+3/2)\,dx\ . \tag{5b}$$

After integration and evaluation, the preceding equations become

$$-2a^2 + 2a = -2b^2 + 2b$$

$$2a^2 - 3a = -b^2 + b/2 - 1/2\ .$$

When solved simultaneously, they yield a feasible solution of $a = 0.271$ and $b = 0.729$. This gives A a share of 39.6% for the subinterval $[0, 0.271]$, B a share of 36.2% for the subinterval $[0.729, 1]$, and C a share of 45.8% for the subinterval $[0.271, 0.729]$.

In fact, however, we can give the worst-off player (B) more without creating envy by squeezing the middle piece further so that C's share (45.8%) decreases as B's share (36.2%) increases (A's share of 39.6% will also increase). Indeed, the middle piece can be reduced to $[a_1, a_2] = [1/3, 2/3]$ while preserving envy-freeness, which is when there is a tie for C between the diminished middle piece and the enlarged left and right pieces.

At these cutpoints, C's share is 33.3%, B's share 38.9%, and A's share 44.4%, which is the envy-free allocation farthest from being maximin. In effect, squeezing the middle piece until it ties for C with the left and right pieces carries the squeezing process too far to produce a maximin envy-free allocation, just as squeezing this piece until it ties for B is not far enough; the maximin envy-free allocation lies in between.

In Example 1, because it is B who receives the smallest share (36.2%) from the envy-free allocation, the referee needs to increase its share. He or she does so by further squeezing the squeezed player's (C's) allocation until it ties with B's while preserving envy-freeness. More specifically, the referee ensures that the squeezer (A) equally values the enlarged pieces on the left and right, using Equation (6a)—which is the same as

Equation (5a)—and ensuring that B's left share and C's middle share—in their respective measures—are equal, using Equation (6b):

$$\int_0^a (-4x+2)\,dx = \int_b^1 (4x-2)\,dx \tag{6a}$$

$$\int_0^a (-2x+3/2)dx = \int_a^{1/2} dx + \int_{1/2}^b dx\,. \tag{6b}$$

After integration and evaluation, the preceding equations become

$$-2a^2+2a=-2b^2+2b$$

$$-a^2+5/2a=b\,.$$

When solved simultaneously, they yield a feasible solution of $a = 0.314$ and $b = 0.686$. This gives A a share of 43.1% for the left subinterval [0, 0.314], B a share of 37.2% for the same-length right subinterval [0.686, 1], and C a share of 37.2% for the middle subinterval [0.314, 0.686].

This allocation is envy-free, because C's reduced middle share is still greater than what it perceives A and B receive on the two sides (31.4%). Moreover, A and B, which continue to receive the same-length subintervals, will not be envious of each other or value the middle piece more.

Because this allocation maximizes the minimum share that any player receives, it is the maximin envy-free allocation. Squeezing C's middle piece further would reduce C's share below 37.2% while increasing A's and B's shares, preventing the envy-free allocation from being maximin.

Recall that an allocation is maximally equitable when all players receive equally valued shares, as each player values its share, that are maximal. As shown in Brams, Jones, and Klamler (2006), when A's share is set equal to C's, and C's share it set equal to B's (in each player's measure), this share in Example 1 is 39.3%. But this allocation is not envy-free, because it requires that a be closer to 0 (at $x = 0.269$) than b is to 1 (at $x = 0.662$). The horizontal length of C's piece ($0.662 - 0.269 = 0.393$) is exactly the common share that all players receive.

We summarize in the table below the subintervals, and the percent shares, that each player receives according to the different algorithms:

	A	B	C
Envy-free allocation	[0, 0.271] 39.6%	[0.729, 1] 36.2%	[0.271, 0.729] 45.8%
Envy-free allocation (maximin)	[0, 0.314] 43.1%	[0.686, 1] 37.2%	[0.314, 0. 686] 37.2%
Equitable allocation (maximal)	[0, 0.269] 39.3%	[0.662, 1) 39.3%	[0.269, 0.662] 39.3%

The maximin envy-free allocation only slightly raises the minimum value that a player receives from 36.2% to 37.2% (for *B*) while lowering *C*'s share by 8.6 percentage points and raising *A*'s share by 3.5 percentage points. Thus, the biggest effect of ensuring that the envy-free allocation is maximin is not on the minimum share that the worst-off player receives but on the shares of the other players, benefiting one player (*A*) and hurting the other (*C*).

Example 2. Modify Example 1 by replacing player *C* with player *D*, whose value function is $v_D(x) = -2x + 2$ over [0,1]. Thus, *D* has a downward sloping value function, most valuing cake near $x = 0$ and least valuing it near $x = 1$, as shown in Figure 2.

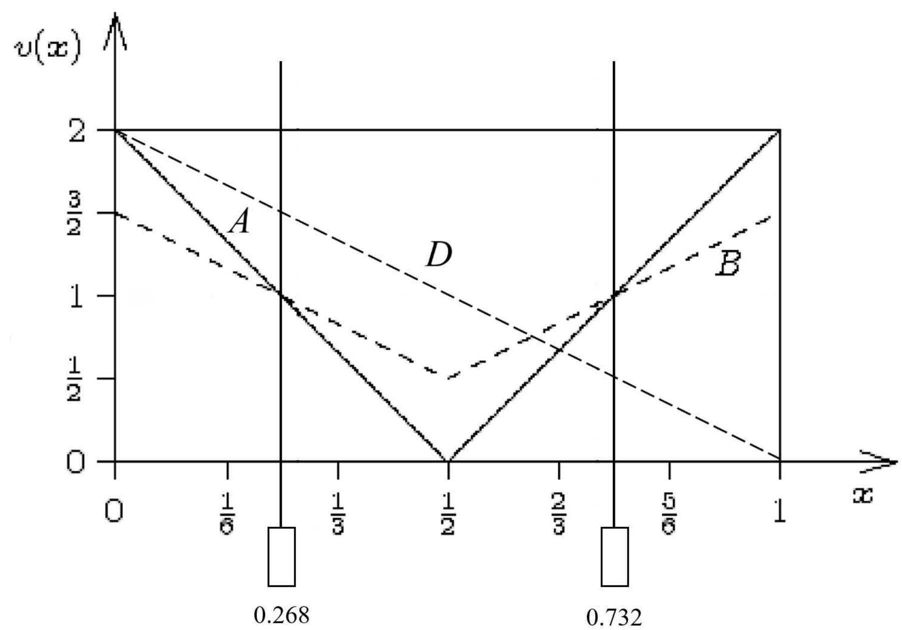

Figure 2. Value Functions *v*(*x*) of *A*, *B*, and *C* in Example 2 and Maximin Envy-Free Cutpoints

The 1/3 and 2/3 marks of the three players are as follows (though not to scale):

0-----$\underline{d_1}$-----a_1-----b_1------------------$\underline{d_2}$-----b_2-----a_2-----1.

Because d_1 precedes a_1 and b_1 on the left, D (instead of A in Example 1), whose 1/3 mark is closest to 0, would initiate the algorithm. From the underscored marks of the players, however, it is unclear whether A, B, or both more value the middle piece $[d_1, d_2]$ or the right piece $[d_2, 1]$.

To determine whether A, B, or both players prefer (before squeezing) the middle piece $[d_1, d_2]$ or the right piece $[d_2, 1]$ when D trisects the cake, we calculate the marks d_1 and d_2 that define this piece for D:

$$\int_0^{d_1} (-2x+2)\,dx = 1/3 \text{ and } \int_{d_2}^{1} (-2x+2)\,dx = 1/3.$$

The left equation gives $d_1 = 0.184$, and the right equation gives $d_2 = 0.423$. Manifestly, both A and B prefer the right subinterval [0.423, 1], which gives each player more than 50% of the total value of the cake, to the middle subinterval [0.184, 0.423]. Hence, Step 5 of the algorithm is applicable, in which C squeezes the right piece from the left and the middle so that it ties for A or B with the enlarged middle or right pieces.

Enlarging the left piece [0, 0.184] and the middle piece [0.184, 0.423] equally requires that these two pieces be equal for D:

$$\int_0^{a} (-2x+2)dx = \int_a^{b} (-2x+2)dx$$

$$-2a^2 + 4a = -b^2 + 2b\,. \tag{7a}$$

D will stop squeezing whichever of the following two events occurs first:

• The enlarged left piece, which A values more than B, equals the diminished right piece for A:

$$\int_0^{a} (-4x+2)\,dx = \int_b^{1} (4x-2)\,dx$$

$$-2a^2 + 2a = -2b^2 + 2b\,. \tag{7b}$$

Solving simultaneously (7a) and (7b) yields a feasible solution of $a = 0.268$ and $b = 0.732$. This gives B a share of 34.0% for the subinterval [0, 0.236]—when a tie for A occurs with the enlarged left piece—D a share of 46.4% for the subinterval [0.268, 0.732], and A a share of 39.2% for the subinterval [0.732, 1]. Hence, when a tie occurs for A with the diminished right piece, the players receive pieces in the order A-D-B, going from left to right.

• The enlarged middle piece, which B values more than A, equals the diminished right piece for B:

$$\int_a^{1/2}(-2x+3/2)\,dx+\int_{1/2}^{b}(2x-1/2)\,dx=\int_b^1(2x-1/2)\,dx$$

$$a^2-3/2a=-2b^2+b\,. \tag{7c}$$

Solving simultaneously (7a) and (7c) yields a feasible solution of $a = 0.267$ and $b = 0.727$. This gives D a share of 46.3% for the subinterval [0, 0.267]—when a tie for B occurs with the enlarged middle piece—B a share of 33.4% for the subinterval [0.267, 0.727], and A a share of 39.7% for the subinterval [0.727, 1]. Hence, when a tie occurs for B, the players receive pieces in the order *D-B-A*.

Because the *D-B-A* solution occurs slightly before (at $a = 0.267$) the *A-D-B* solution (at $a = 0.268$), the envy-free algorithm gives the *D-B-A* solution, whereby D gets the left piece, where its value is most concentrated. If the squeezing of the right piece continues to $a = 0.268$, the right piece will be slightly reduced—giving D 46.3% rather than 46.4%—so that it ties with the middle piece for B, giving the *A-D-B* solution, in which D gets the middle piece.

We know that the latter solution is envy-free for A and B, because these players get left and right pieces, respectively, that are equidistant from the endpoints of 0 and 1. Moreover, D will not be envious of either A or B, because, as squeezer, D maintains a tie between the middle piece and left piece, both of which D values more than the diminished right piece.

Between the two envy-free solutions of *D-B-A* and *A-D-B*, *A-D-B* gives the worst-off player (A) a bit more (34.0%) than the *D-B-A* solution gives the worst-off player (33.4% for B), so *A-D-B* is the maximal envy-free solution. Unlike Example 1, the *D-B-A* solution does not give A and B equal-length pieces, whereas the *A-D-B* solution does.

If the order of players from left to right is *A-D-B*, there is an equitable solution, which can be found by setting A's valuation of its piece equal to B's and requiring that D value A's piece the same as its piece:

$$\int_0^a(-4x+2)\,dx=\int_0^a(2x-1/2)\,dx$$

$$\int_0^a(-2x+2)\,dx=\int_a^b(-2x+2)\,dx\,.$$

After integration and evaluation, the preceding equations become

$$-a^2 + 2a = -2b^2 + 2b$$

$$-2a^2 + 4a = -b^2 + 2b\ .$$

When solved simultaneously, they yield a feasible solution of $a = 0.256$ and $b = 0.675$, with *A* getting the left piece, *D* the middle piece, and *B* the right piece. This gives each player a share of 38.2%, but it is not an envy-free allocation, because *A* and *B* receive different-length pieces.

By comparison, if the left-to-right order of the players is *D*-*B*-*A*, there is a different equitable solution, which can be found by setting *D*'s valuation of its piece equal to *A*'s and requiring that *B* value *A*'s piece the same as its piece:

$$\int_0^a (-2x+2)\,dx = \int_b^1 (4x-2)\,dx$$

$$\int_a^{1/2} (-2x+3/2)\,dx + \int_{1/2}^b (2x-1/2)\,dx = \int_b^1 (4x-2)\,dx\ .$$

After integration and evaluation, the preceding equations become

$$-a^2 + 2a = -2b^2 + 2b$$

$$a^2 - 3/2a + 1/2 = -3b^2 + 5/2b\ .$$

When solved simultaneously, they yield a feasible solution of $a = 0.219$ and $b = 0.734$, with *D* getting the left piece, *B* the middle piece, and *A* the right piece. This gives each player a share of 39.0%, but it is not an envy-free allocation, because *D* envies *B* for getting a piece that *D* values at 53.9%.

We summarize in the table that follows the subintervals, and the percent shares, that each player receives according to the algorithms and the other properties that we have discussed:

	A	*B*	*D*
Envy-free allocation *Left-to-Right Order: D-B-A*	[0.727, 1] 39.7%	[0.267, 0.727] 33.4%	[0, 0.267] 46.3%
Envy-free allocation (maximin) *Left-to-Right Order: A-D-B*	[0, 0.268] 39.2%	[0.732, 1] 34.0%	[0.268, 0. 732] 46.4%
Equitable allocation *Left-to-Right Order: A-D-B*	[0, 0.256] 38.2%	[0.675, 1] 38.2%	[0.256, 0.675] 38.2%
Equitable allocation (maximal) *Left-to-Right Order: D-B-A*	[0.734, 1] 39.0%	[0.219, 0.734] 39.0%	[0, 0.219] 39.0%

Unlike Example 1, the order from left to right in which the players receive pieces makes a difference for both the envy-free and the equitable allocations.[4] Whereas both orders of the two envy-free allocations are efficient, this is not so for the equitable allocations: The order *D*-*B*-*A* Pareto-dominates the order *A*-*D*-*B*, making *D*-*B*-*A* the maximally equitable allocation. On the other hand, the order *A*-*D*-*B* is the maximin envy-free allocation, because the minimum it gives to a player (34.0% to *A*) is greater than the minimum that *D*-*B*-*A* gives to a player (33.4% to *B*).

5. Conclusions

We have described a 3-person, 2-cut algorithm for finding an envy-free allocation of a cake—inspired by the continuous moving-knife procedure of Barbanel and Brams (2004)—when the players submit their value functions to a referee instead of moving knives. The allocation is efficient and gives connected pieces, which we illustrated with two examples when the players' value functions are piecewise linear.

But more than duplicating the results of a moving knife procedure, the players' submission of their value functions enables the referee to make calculations, which are purely mechanical, that yield a maximin envy-free allocation. In general, this raises the value that the worst-off player receives, which makes it arguably fairer than the Barbanel-Brams (2004) envy-free allocation.[5] Similarly, a referee can use the value functions to find a maximally equitable allocation, which gives all players equal shares—as they value them—that are maximal. While the latter allocation may raise the value of the piece that the worst-off player receives from the maximin envy-free allocation, it may do so at the price of creating envy.

It is not evident that the envy-free algorithms can be extended to $n > 3$ players and still use the minimal n–1 cuts. In particular, the extensions present two hurdles: Finding the order of players along a line to which to assign a portion of the cake, and determining the cutpoints that give them envy-free pieces.

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[4] Another difference is that the maximin envy-free allocation in Example 1 occurs when the value to the worst-off player (*B*) of its piece (37.2%) equals the value to the recipient (*C*) of the squeezed piece. In Example 2, the maximin envy-free allocation does not equalize these values, because it is based on a different order (*A*-*D*-*B*) from the Barbanel-Brams (2004) envy-free allocation (*D*-*B*-*A*), which creates a tie earlier (when $a = 0.267$).

[5] This is also true of the maximin envy-free division of indivisible items (Brams, Kilgour, and Klamler, 2017; Kurokawa, Procaccia, and Wang, 2018); for empirical evidence in support of maximin, see Gal, Mash, Procaccia, and Zick (2018).

References

Amanatidis, Georgios, George Christodoulou, John Fearnley, Evangelos Markakis, Christos-Alexandros Psomas, and Etftychia Vakaliou (2018). "An Improved Envy-Free Cake Cutting Protocol for Four Agents." Preprint. http://www.cs.cmu.edu/~cpsomas/envy_free.pdf

Aziz, Haris, and Simon Mackenzie (2017). "A Discrete and Bounded Envy-Free Cake Cutting Protocol for Any Number of Agents." Preprint. https://arxiv.org/abs/1604.03655

Barbanel, Julius B. (2005). *The Geometry of Efficient Fair Division*. New York: Cambridge University Press.

Barbanel, Julius B., and Steven J. Brams (2004). "Cake Division with Minimal Cuts: Envy-Free Procedures for 3 Persons, 4 Persons, and Beyond." *Mathematical Social Sciences* 48, no. 3 (November): 251-259.

Brams, Steven J., Michael A. Jones, and Christian Klamler (2006). "Better Ways to Cut a Cake." *Notices of the AMS* 35, no. 11 (December): 1314-1321.

Brams, Steven J., D. Marc Kilgour, and Christian Klamler (2017). "Maximin Envy-Free Division of Indivisible Items," *Group Decision and Negotiation* 26, no. 1 (January): 115-131.

Brams, Steven J., and Alan D. Taylor (1995). "An Envy-Free Cake Division Protocol." *American Mathematical Monthly* 102, no. 1 (January): 9-18.

Brams, Steven J., and Alan D. Taylor (1996). *Fair Division: From Cake-Cutting to Dispute Resolution*. New York: Cambridge University Press.

Brams, Steven J., Alan D. Taylor, and William S. Zwicker (1997). "A Moving-Knife Solution to the Four-Person Envy-Free Cake Division Problem." *Proceedings of the American Mathematical Society* 125, no. 2 (February): 547-554.

Dehghani, Sina, Alireza Farhadi, MohammadTaghi HajiAghayi, and Hadi Yami (2018). "Envy-Free Chore Division for an Arbitrary Number of Agents." *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*. http://epubs.siam.org/doi/abs/10.1137/1.9781611975031.164

Gal Ya'akov (Kobi), Moshe Mash, Ariel D. Procaccia, and Yair Zick (2018), "Which is the Fairest (Rent Division) of Them All?" *Journal of the ACM* (forthcoming).

Gale, David (1993). "Mathematical Entertainments." *Mathematical Intelligencer* 15, no. 1 (Winter): 48-52.

Ianovski, Egor (2012). "Cake Cutting Mechanisms." Preprint. https://arxiv.org/pdf/1203.0100.pdf

Kurokawa, David, Ariel D. Procacia, and Junxing Wang (2018). "Fair Enough: Guaranteeing Approximate Maximin Shares." *Journal of the ACM* 65, no. 2: 8:1-8:27.

Neyman, Jerzy (1946). "Un Théorèm d'Existence." *C. R. Academie de Science Paris* 222: 843-845.

Stromquist, Walter (1980). "How to Cut a Cake Fairly." *American Mathematical Monthly* 18, no. 8 (October): 640-644.

Stromquist, Walter (2008). "Envy-Free Cake Divisions Cannot Be Found by Finite Protocols." *Electronic Journal of Combinatorics* 15, #R11. http://www.combinatorics.org/ojs/index.php/eljc/article/view/v15i1r11/pdf

Su, Francis Edward (1999). "Rental Harmony: Sperner's Lemma in Fair Division." *American Mathematical Monthly* 106, no. 10 (December): 430-442.

Wolfram Equation Solver (2018). https://www.wolframalpha.com/examples/EquationSolving.html